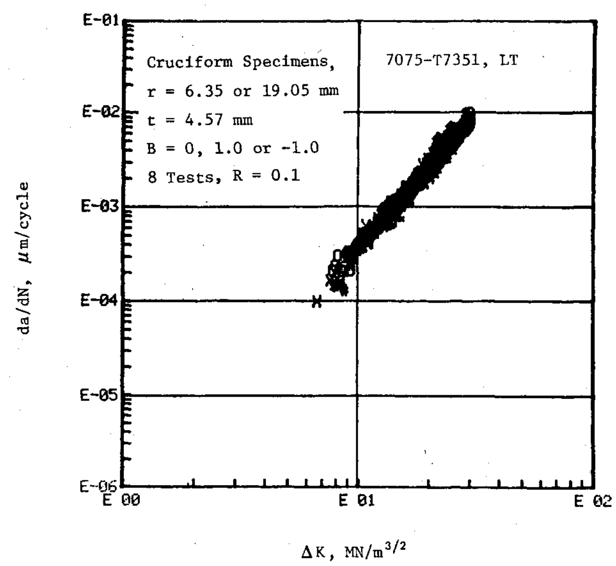

Fig. 2 Cyclic crack growth behavior for cracks at a hole.

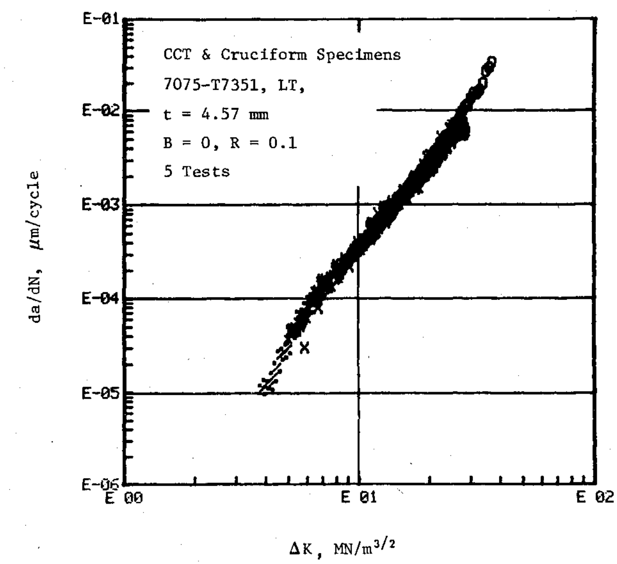

Fig. 3 Cyclic crack growth behavior of center cracked specimen.

diameter. Each specimen was cyclically precracked from a crack starter (a very small saw cut approximately 1.0 mm in length) at both sides of the hole. The stress amplitude was 82.68 MPa and 8.268 MPa for σ_y with B being either 0, +1, or −1. A crack growth history for each test was recorded. The K value for each data point was computed using the superposition technique of Eq. (3). According to Ref. 2, F was equal to unity for the cruciform specimen. All da/dN vs ΔK data points, for all eight tests at three different biaxial stress ratios, are presented in Fig. 2. A group of data points representing the baseline crack growth rate behavior of the same material (i.e., developed from specimens without a hole) is presented in Fig. 3. It is seen that the crack growth rate behavior in all seven cases (a big hole or a small hole in either one of the three biaxial loading conditions, and a baseline condition, i.e., $\sigma_x = 0$, no hole) are the same. Therefore the result of this comparison has indicated that stress intensity factors for cracks at a hole can be accurately computed by using Eq. (3).

In conclusion, it has been verified by experiments that lateral stresses parallel to the crack may or may not affect the crack tip stress intensity factor depending upon specimen geometry and crack morphology. As stated earlier, wherever there is no geometric complexity, e.g., a straight crack in a semi-infinite sheet, the K values will be the same for all the cases, i.e., with and without σ_x. Otherwise, e.g., cracks at the edge of a hole, the K-factor for a given crack will change from one biaxial ratio to another biaxial ratio. However, the crack growth rate at a given ΔK level will be the same for all the cases as long as the K value for each case is properly determined.

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Finite Elements for Generalized Plane Strain

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Introduction

A SURVEY of the available textbooks on finite elements leads to the surprising observation that most authors omit even a passing reference to generalized plane strain analysis of a long, linearly elastic cylinder for which the axial force is specified. Apparently the only exception is Gallagher,[1] who proposes that the axial strain be treated as an unknown and be calculated, just like the nodal variables for the mesh, by solving the assembled stiffness equations.

Gallagher's approach is direct, clear, and theoretically elegant, but in some practical situations it has two drawbacks. First, because the axial strain directly influences the stress state in *every* element in the mesh, the stiffness matrix is not banded (the axial strain unknown appears in every equation of the assembled system). Because the stiffness matrix, even if not banded, is still sparse, this objection can be met to some extent by using an efficient equation solver (e.g., a frontal solver), which avoids unnecessary operation with zeroes. Nevertheless, the suspicion remains that there should be a way to formulate the problem so that this source of computational inefficiency is avoided altogether.

The second practical drawback to treating the axial strain as an additional unknown (like an additional nodal variable) is that implementing this approach in an existing computer program requires many conceptually trivial, but in practice, time-consuming and error-prone coding changes. The presence of "one more" variable means that indexing arrays

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and counting schemes in many parts of the program must be modified. The required changes to the program are not easily isolated in separate routines, but occur throughout the program.

To meet these objections, in the present Note, a method is described which preserves the bandedness of the stiffness matrix and which can be implemented in an existing program with minimum disturbance to the code. Indeed, if an existing plane strain program *cannot* be modified (e.g., because it is available through a computer service bureau), the method can be used to prepare special input data and combine results in such a way that generalized plane strain problems with specified axial force can still be solved, even though the program was not originally designed for such problems.

The method to be described is a generalization of the well-known analytical technique of classical linear elasticity wherein the (trivial) solution corresponding to an axial force acting alone is determined by inspection and then superposed with the solution corresponding to in-plane forces acting alone. Unlike this classical technique, however, the present method is applicable even if the in-plane displacement boundary conditions are such that the "axial force solution" is non-trivial and thus *cannot* be determined by inspection.

Problem Formulation

Consider a long prismatic cylinder under conditions of generalized plane strain. That is, the transverse force and displacement boundary conditions do not vary along the length of the cylinder, and the axial strain has the same value everywhere. Let the z axis lie parallel to the axis of the cylinder, and let the x and y axes lie in a plane perpendicular to the z axis. Let A represent the cross-sectional area of the cylinder. For a linearly elastic isotropic or orthotropic material, the normal stress in the axial direction can be calculated from the generalized Hooke's law,

$$\sigma_z = C_{zx}\epsilon_x + C_{zy}\epsilon_y + C_{zz}\epsilon_z \tag{1}$$

where C_{zx}, C_{zy}, and C_{zz} are elastic constants, and ϵ_x, ϵ_y, and ϵ_z are normal strains in the coordinate directions. The normal stress σ_z is related to the specified axial force F by the equation

$$\int_A \sigma_z \mathrm{d}A = F \tag{2}$$

Following the procedure described briefly by Gallagher,[1] we can derive the finite element version of the equilibrium equations for generalized plane strain in terms of an array of in-plane nodal variables, δ, and the (unknown) uniform axial strain, ϵ. Collection of the stiffness coefficients associated with ϵ in a column vector, g, and transposition of this vector to the right-hand side leads to the following form of the equilibrium equations:

$$K\delta = f - \epsilon g \tag{3}$$

in which K is the global stiffness matrix for plane strain (*not generalized* plane strain) and f is the load vector arising from in-plane forces. The particular scalar equilibrium equation corresponding to the axial force is not included in Eq. (3) because it is implied by Eqs. (1) and (2). Thus, a solution to the generalized plane strain problem consists of an array δ and a scalar ϵ satisfying equilibrium, Eq. (3), and the axial force condition specified through Eqs. (1) and (2) with $\epsilon = \epsilon_z$. A solution can be found through the following six-step procedure.

Solution Procedure

1) Factor the stiffness matrix K using for example the Cholesky decomposition technique.

2) Solve for δ^f from the relation

$$K\delta^f = f \tag{4}$$

using forward elimination and back substitution.

3) The in-plane strains ϵ_x^f and ϵ_y^f corresponding to the nodal displacements δ^f now can be evaluated at any point in the domain. Use these strains to calculate

$$F^f = \int_A (C_{zx}\epsilon_x^f + C_{zy}\epsilon_y^f)\,\mathrm{d}A \tag{5}$$

4) Repeat steps 2 and 3 with f replaced by g.

5) Calculate the axial strain,

$$\epsilon = (F - F^f) / (AC_{zz} - F^g) \tag{6}$$

where F is the specified axial force [Eq. (2)].

6) Calculate the final result,

$$\delta = \delta^f - \epsilon\,\delta^g \tag{7}$$

Justification of the Method

That the nodal displacements δ do in fact satisfy the equilibrium equations is easily shown. Multiply Eq. (7) by K and use Eq. (4) twice (once with f and once with g) to get

$$K\delta = K\delta^f - \epsilon K\delta^g = f - \epsilon g \tag{8}$$

Thus, the nodal displacements δ correspond to in-plane loads specified by the load vector f and an imposed axial strain ϵ given by Eq. (6). We must still show that this imposed axial strain will produce an axial force equal to the specified value F. To show this fact, note first that because of the linearity of the strain-displacement relations and Eq. (7), the in-plane normal strains calculated from δ are related to those calculated from δ^f and δ^g by

$$\epsilon_x = \epsilon_x^f - \epsilon\epsilon_x^g \text{ and } \epsilon_y = \epsilon_y^f - \epsilon\epsilon_y^g \tag{9}$$

The axial force calculated from the nodal displacements δ is then, by Eq. (1),

$$\begin{aligned}\int_A \sigma_z \mathrm{d}A &= \int_A (C_{zx}\epsilon_x + C_{zy}\epsilon_y + C_{zz}\epsilon_z)\,\mathrm{d}A \\ &= \int_A (C_{zx}\epsilon_x^f + C_{zy}\epsilon_y^f)\,\mathrm{d}A - \epsilon\int_A (C_{zx}\epsilon_x^g + C_{zy}\epsilon_y^g)\,\mathrm{d}A \\ &\quad + \int_A C_{zz}\epsilon_z \mathrm{d}A = F^f - \epsilon F^g + C_{zz}\epsilon A\end{aligned} \tag{10}$$

where the last line follows from Eq. (5) and the observation that $\epsilon_z = \epsilon$. Substituting the expression for ϵ [Eq. (6)] into the right-hand side now yields Eq. (2).

Concluding Remarks

Some final observations on the method can now be made. First, note that for the superposition expressed by Eq. (7) to be valid, the in-plane displacement boundary conditions must be homogeneous. Note also that the method depends for its efficiency on the well-known fact that two finite element solutions can be obtained for little more than the cost of one, when Cholesky decomposition is used as in steps 1 and 2.

An observation related to computer implementation of the method is that because most finite element computer programs contain provisions for multiple load vectors, step 2 (for both f and g) can be easily implemented in existing codes. Only step 3 requires significant new programming, and even here the burden might be light, because in many programs, strains are already evaluated at quadrature points—precisely what is needed to evaluate the integral in Eq. (5).

To use this scheme with a commercially available program (which presumably could not be altered), we would have to write a program (or proceed by hand calculation) to generate the array g. Running the commercial program with f and g as separate load cases would then give the strains to be used as input to an additional small, user written program for calculating the force F^f (and F^g) of Eq. (5).

Reference

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Radiative Transfer in Anisotropically Scattering Nonplanar Media

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Nomenclature

B	= emissive power, πI_b
I	= radiation intensity
I_b	= blackbody intensity
ℓ	= direction cosine
n	= radius ratio, r_2/r_1
P	= phase function
Q	= a constant
r	= dimensionless radial direction $(\alpha+\sigma)r'$
r'	= radial direction
X	= position vector
α	= absorption coefficient
β	= polar angle
ϵ	= emissivity
λ	= scattering albedo, $\sigma/(\alpha+\sigma)$
ξ	= $\cos\beta$
σ	= scattering coefficient
τ	= optical thickness, r_2-r_1
ϕ	= azimuthal angle
ω	= solid angle
Ω	= unit direction vector

Subscripts

1,2 = inner and outer boundary, respectively

Introduction

THE exact treatment of radiative transfer in participating, particularly anisotropically scattering media requires immense effort and computation. In the past, several approximate solutions of the equation of transfer have been developed to overcome the mathematical complexity of the problem. The spherical harmonics method (P_N approximation) is capable of estimating higher order approximate solutions. Bayazitoğlu and Higenyi[1,2] employed P_3 and P_5 approximations for nonplanar geometries in problems involving absorbing, emitting, and isotropically scattering media. Their results compared favorably with the existing exact solutions.

The purpose of this Note is to demonstrate, within the P_3 framework, the effects of anisotropic scattering in one-dimensional cylindrical and spherical geometries. To this effect, a linear phase function is used. Modest and Azad[3] showed that a slightly modified form of the linear model, used together with the differential P_1 approximation, yields quite accurate results in planar geometry.

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Analysis

Consider the equation of transfer

$$\Omega\cdot\nabla I=\alpha I_b-(\alpha+\sigma)I+\frac{\sigma}{4\pi}\int_{4\pi}I(\Omega')\cdot p(\Omega,\Omega')\mathrm{d}\omega' \tag{1}$$

The phase function p in Eq. (2) can be expanded in a series of Legendre polynomials[4]:

$$p(\Omega,\Omega')=\sum_{n=0}^{\infty}a_nP_n(\Omega,\Omega'),\ a_0\equiv 1 \tag{2}$$

In the following analysis, only the first two terms in the expansion are retained on grounds of simplicity. However, the spherical harmonics method can accommodate any number of terms.

In the spherical harmonics method, the angular distribution of the radiation intensity is expressed in a series of associated Legendre polynomials. The series is truncated after N terms (P_N approximation), and the coefficients are expressed in terms of the moments of intensity. In this respect, the extension of the method to anisotropic scattering problems is straightforward. When the phase function is expressed in the form of Eq. (2), the scattering integral in the equation of transfer is transformed into a summation term in the moments of intensity, thus enabling the complex anisotropic scattering effects to be represented and handled in a simple and efficient manner.

In the final step, the equation of transfer is approximated by a series of "moment" differential equations which are derived by multiplying it by the powers of direction cosines and integrating over a solid angle of 4π.

To demonstrate the effects of anisotropic scattering in one-dimensional cylindrical and spherical geometrics, a gray medium at radiative equilibrium is considered.

Cylindrical Geometry (r,θ,z)

For cylindrical symmetry, the equation of transfer is

$$\ell_r\frac{\partial I}{\partial r}-\ell_\theta\frac{\partial I}{\partial\phi}=(1-\lambda)I_b-I+\frac{\lambda}{4\pi}\int_{4\pi}(I+a_1\ell_r)I\mathrm{d}\omega \tag{3}$$

where $\ell_r=\sin\beta\cos\phi$ and $\ell_\theta=\sin\beta\sin\phi$. After some algebraic manipulation, the moment differential equations take the form:

$$\frac{\mathrm{d}I_0}{\mathrm{d}r}=\frac{10}{3r}I_0-(10-\lambda a)I_r-\frac{10}{r}I_{rr}+\frac{35}{3}I_{rrr},\quad a\equiv a_1 \tag{4a}$$

$$\frac{\mathrm{d}I_r}{\mathrm{d}r}=(1-\lambda)(4B-I_0)-\frac{1}{r}I_r \tag{4b}$$

$$\frac{\mathrm{d}I_{rr}}{\mathrm{d}r}=\frac{2}{3r}I_0-\left(1-\frac{\lambda a}{3}\right)I_r-\frac{2}{r}I_{rr} \tag{4c}$$

$$\frac{\mathrm{d}I_{rrr}}{\mathrm{d}r}=\frac{(1-\lambda)}{3}4B+\frac{\lambda}{3}I_0+\frac{8}{5r}I_r-I_{rr}-\frac{3}{r}I_{rrr} \tag{4d}$$

where I_0 is the zeroth moment of intensity, and I_r, I_{rr}, and I_{rrr} are the first, second, and third moments of intensity in the radial direction. Radiative equilibrium demands $I_r=Q/r$ and therefore from Eq. (4b), $I_0=4B$.